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# On the exact propagator beyond and at caustics

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**Abstract.** We evaluate exactly the propagator beyond and at caustics for a harmonically bound electron subject to a constant magnetic field and a time-dependent electric field. Our new results are confirmed by investigating the classical paths joining two end-point positions.

For a harmonically bound electron subject to a constant magnetic field $\boldsymbol{B}$ (along $z$ direction) and a time-dependent electric field $\boldsymbol{E}(t)$ (in $xy$ plane), the Lagrangian has the form

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = L_{\parallel}(z, \dot{z}, t) + L_{\perp}(\boldsymbol{r}_{\perp}, \dot{\boldsymbol{r}}_{\perp}, t) \tag{1}$$

with

$$L_{\parallel}(z, \dot{z}, t) = \tfrac{1}{2}m(\dot{z}^2 - \Omega^2 z^2) + eE_z(t) \tag{2}$$

$$L_{\perp}(\boldsymbol{r}_{\perp}, \dot{\boldsymbol{r}}_{\perp}, t) = \tfrac{1}{2}m(\dot{\boldsymbol{r}}_{\perp}^2 - \Omega^2 \boldsymbol{r}_{\perp}^2 + \omega \tilde{\dot{\boldsymbol{r}}}_{\perp} J \boldsymbol{r}_{\perp}) + e\boldsymbol{E}_{\perp}(t) \cdot \boldsymbol{r}_{\perp} \tag{3}$$

where $\omega = eB/m$ is the cyclotron frequency, $\Omega$ is the oscillator frequency, and $\boldsymbol{r}_{\perp}$ and $\boldsymbol{E}_{\perp}(t)$ denote, respectively, the component of $\boldsymbol{r}$ and $\boldsymbol{E}(t)$ perpendicular to $\boldsymbol{B}$. The $(2\times 2)$ matrix $J$ is given by ($J^2 = -I$, identity matrix)

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Both (2) and (3) are one-time quadratic Lagrangians, and their propagator can be evaluated exactly by using the Van Vleck (1928)–Pauli (1952) method. Jones and Papadopoulos (1971) carried out the calculations and obtained the propagator of Lagrangian (1) as

$$K(\boldsymbol{r}'', \boldsymbol{r}'; T) = K_{\parallel}(z'', z'; T) K_{\perp}(\boldsymbol{r}''_{\perp}, \boldsymbol{r}'_{\perp}; T) \tag{4}$$

with

$$K_{\parallel}(z'', z'; T) = \left(\frac{1}{2\pi i \hbar} \left| \left( \frac{\partial^2 S_{\parallel}(z'', z'; T)}{\partial z'' \partial z'} \right) \right| \right)^{1/2} \exp\left( \frac{i}{\hbar} S_{\parallel}(z'', z'; T) \right)$$

$$= \left( \frac{m\Omega}{(2\pi i \hbar \sin \Omega T)} \right)^{1/2} \exp\left\{ \frac{im\Omega}{2\hbar \sin \Omega T} \left[ (z''^2 + z'^2) \cos \Omega T - 2z'z'' \right.\right.$$

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$$+\frac{2e}{m\Omega}\int_{t'}^{t''}(z'\sin\Omega(t''-t)+z''\sin\Omega(t-t'))E_z(t)\,\mathrm{d}t$$
$$-2\left(\frac{e}{m\Omega}\right)^2\int_{t'}^{t''}\int_{t'}^{t}\sin\Omega(t''-t)\sin\Omega(\tau-t')E_z(t)E_z(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\Bigg]\Bigg\}$$
$$\text{for } 0<T<\pi/\Omega \tag{5}$$

and

$$K_\perp(\boldsymbol{r}''_\perp,\boldsymbol{r}'_\perp;T)=\left(\frac{1}{2\pi\mathrm{i}\hbar}\left|\det\left(\frac{\partial^2 S_\perp(\boldsymbol{r}''_\perp\boldsymbol{r}'_\perp;T)}{\partial\boldsymbol{r}''_\perp\partial\boldsymbol{r}'_\perp}\right)\right|\right)^{1/2}\exp\left(\frac{\mathrm{i}}{\hbar}S_\perp(\boldsymbol{r}''_\perp,\boldsymbol{r}'_\perp;T)\right)$$
$$=\left(\frac{m\Omega'}{2\pi\mathrm{i}\hbar\sin\Omega'T}\right)\exp\Bigg\{\frac{\mathrm{i}m\Omega'}{2\hbar\sin\Omega'T}$$
$$\times\Bigg[(\boldsymbol{r}''^2_\perp+\boldsymbol{r}'^2_\perp)\cos\Omega'T-2\boldsymbol{r}'_\perp\,\mathrm{e}^{J\omega T/2}\boldsymbol{r}''_\perp$$
$$+\frac{2e}{m\Omega'}\int_{t'}^{t''}[\sin\Omega'(t''-t)\tilde{\boldsymbol{r}}'\,\mathrm{e}^{-J\omega t'/2}$$
$$+\sin\Omega'(t-t')\boldsymbol{r}''_\perp\,\mathrm{e}^{-J\omega t''/2}]\,\mathrm{e}^{J\omega t/2}\boldsymbol{E}_\perp(t)\,\mathrm{d}t$$
$$-2\left(\frac{e}{m\Omega'}\right)^2\int_{t'}^{t''}\int_{t'}^{t}\sin\Omega'(t''-t)\sin\Omega'(\tau-t')$$
$$\times\tilde{\boldsymbol{E}}_\perp(t)\,\mathrm{e}^{-J\omega(t-\tau)/2}\boldsymbol{E}_\perp(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\Bigg]\Bigg\} \tag{6}$$

for $0<T<\pi/\Omega'$. Here we have assumed that $\Omega'^2=\Omega^2+\omega^2/4$, $T=t''-t'$ and $(\boldsymbol{r}''_\perp, t'')$ and $(\boldsymbol{r}'_\perp, t')$ are the final and initial spacetime points. We have also defined

$$\mathrm{e}^{\pm J\theta}=\cos(\theta)I\pm\sin(\theta)J$$

for any variable $\theta$.

However the propagator (4) obtained is only valid for $0<T<\pi/\Omega'$. In this letter we have extended their results to the following cases: (*a*) beyond caustics, $\sin\Omega T\neq 0$ with $\Omega T>\pi$ or $\sin\Omega' T\neq 0$ with $\Omega' T>\pi$ and (*b*) at caustics, $\sin\Omega T=0$ or $\sin\Omega' T=0$. Since the $z$ coordinate can always be separated from $\boldsymbol{r}_\perp$ ($x$ and $y$) coordinates, we can evaluate $K_\parallel(z'', z'; T)$ and $K_\perp(\boldsymbol{r}''_\perp, \boldsymbol{r}'_\perp; T)$ beyond and at caustics, independently.

By extending the Feynman formula (Feynman and Hibbs 1965), Horváthy (1979) obtained the propagator of the harmonic oscillator by including the Maslov correction factor. His result has been generalised by Cheng (1984a) for the time-dependent forced harmonic oscillator. Thus we have

$$K_\parallel(z'',z';T>\pi/\Omega)=\left(\frac{m\Omega}{2\pi\hbar|\sin\Omega T|}\right)^{1/2}M_\Omega\exp\left(\frac{\mathrm{i}}{\hbar}S_\parallel(z'',z';T)\right) \tag{7}$$

with

$$M_\Omega=\exp\{-\mathrm{i}\pi[1+2\,\mathrm{ent}(\Omega T/\pi)]/4\}$$

and

$$K_{\|}(z'', z';\ T=k\pi/\Omega)=\exp(-\mathrm{i}k\pi/2)\exp\left[\frac{\mathrm{i}e}{\hbar}\left(z''\int_{t'}^{t''}\cos\Omega(t''-t)E_z(t)\,\mathrm{d}t\right.\right.$$
$$\left.\left.-\frac{e}{m\Omega}\int_{t'}^{t''}\int_{t'}^{t}\sin\Omega(t''-t)\cos\Omega(t''-\tau)E_z(t)E_z(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\right)\right]$$
$$\times\delta(z''-z_p''-(-1)^k z')\qquad (k \text{ being a positive integer}) \tag{8}$$

with

$$z_p''=(e/m\Omega)\int_{t'}^{t''}\sin\Omega(t''-t)E_z(t)\,\mathrm{d}t.$$

Here $\mathrm{ent}(\Omega T/\pi)$ stands for the greatest integer which is less than or equal to $\Omega T/\pi$.

Following the idea of Levit and Smilansky (1977) by expanding the path variations in terms of the basis of the eigenfunctions in Morse's boundary problem (Morse 1934), we then obtain from the Lagrangian (3)

$$\begin{aligned} m\ddot{\eta}_1-\tfrac{1}{2}m\omega\dot{\eta}_2+(m\Omega^2+\lambda)\eta_1=0\\ m\ddot{\eta}_2+\tfrac{1}{2}m\omega\dot{\eta}_1+(m\Omega^2+\lambda)\eta_2=0\end{aligned} \tag{9}$$

with the boundary conditions

$$\eta_1'=\eta_2'=\eta_1''=\eta_2''=0$$

for the path variations $\boldsymbol{\eta}(t)$, which is defined as $\boldsymbol{\eta}(t)=\boldsymbol{r}_\perp(t)-\boldsymbol{r}_\perp^{\mathrm{cl}}(t)$, and $\boldsymbol{r}_\perp^{\mathrm{cl}}(t)$ is the classical path connecting $(\boldsymbol{r}_\perp', t')$ and $(\boldsymbol{r}_\perp'', t'')$. It can easily find the eigenvalues of (9),

$$\lambda^{(n)}=m(n^2\pi^2/T^2)[1-(\Omega' T/n\pi)^2]\qquad n=1,2,\ldots \tag{10}$$

which are double degenerate. By studying (10) carefully, the index of the classical path is $2\mathrm{ent}(\Omega' T/\pi)$. Here $\mathrm{ent}(\Omega' T/\pi)$ represents the greatest integer which is less than or equal to $\Omega' T/\pi$. In other words, the Morse theory (Milnor 1963) can be used to present the phase of the propagator in terms of the number of focal points on the classical path between the initial and final positions.

Hence we have

$$K_\perp(\boldsymbol{r}_\perp'', \boldsymbol{r}_\perp';\ T>\pi/\Omega')=\left(\frac{m\Omega'}{2\pi\hbar|\sin\Omega' T|}\right)M_{\Omega'}\exp\left(\frac{\mathrm{i}}{\hbar}S_\perp(\boldsymbol{r}_\perp'', \boldsymbol{r}_\perp';\ T)\right) \tag{11}$$

with

$$M_{\Omega'}=\exp\{-\mathrm{i}\pi[1+2\mathrm{ent}(\Omega' T/\pi)]/2\}$$

since the time-dependent electric field $\boldsymbol{E}_\perp(t)$ will not affect the frequency $\Omega'$. Now we introduce the modified semi-group property of the propagator (Cheng 1984b)

$$K_\perp(\boldsymbol{r}_\perp'', \boldsymbol{r}_\perp';\ T=j\pi/\Omega')=\exp\left(-\frac{\mathrm{i}j\pi}{2}\right)|K_\perp(\boldsymbol{r}_\perp'', \boldsymbol{r}_\perp;\ t''-t)||K_\perp(\boldsymbol{r}_\perp, \boldsymbol{r}_\perp';\ t-t')|$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(\frac{\mathrm{i}}{\hbar}[S_\perp(\boldsymbol{r}_\perp'', \boldsymbol{r}_\perp;\ t''-t)+S_\perp(\boldsymbol{r}_\perp, \boldsymbol{r}_\perp';\ t-t')]\right)\mathrm{d}\boldsymbol{r}_\perp$$
$$j=1,2,\ldots \tag{12}$$

since the Maslov correction jumps in phase at every half-period at caustics. However, we should mention that in evaluating (12) one must choose $t$ so that there exists one and only one classical path between the pair of spacetime points $(\boldsymbol{r}'_\perp, t')$ and $(\boldsymbol{r}_\perp, t)$, and $(\boldsymbol{r}_\perp, t)$ and $(\boldsymbol{r}''_\perp, t'')$ considered. Choosing $\Omega'(t''-t)=\pi/2$ and carrying out the integration (12) with the help of (11), we arrive at

$$
\begin{aligned}
&K_\perp(\boldsymbol{r}''_\perp, \boldsymbol{r}'_\perp; T=j\pi/\Omega') \\
&\quad = \exp\left(-\frac{\mathrm{i}j\pi}{2}\right)\exp\left[\frac{\mathrm{i}e}{\hbar}\left(\boldsymbol{r}''_\perp \int_{t'}^{t''} \cos\Omega'(t''-t)\,\mathrm{e}^{-J\omega(t''-t)/2}\boldsymbol{E}_\perp(t)\,\mathrm{d}t\right.\right. \\
&\qquad -\frac{e}{m\Omega'}\int_t^{t''}\int_{t'}^{t} \sin\Omega'(t''-t)\cos\Omega'(t''-\tau) \\
&\qquad \left.\left.\times \boldsymbol{E}_\perp(t)\,\mathrm{e}^{-J\omega(t-\tau)/2}\boldsymbol{E}_\perp(\tau)\,\mathrm{d}t\,\mathrm{d}\tau\right)\right] \\
&\qquad \times \delta^{(2)}(\boldsymbol{r}''_\perp - \boldsymbol{R}''_p - (-1)^j\,\mathrm{e}^{-J\omega T/2}\boldsymbol{r}'_\perp)
\end{aligned}
\tag{13}
$$

after lengthy but straightforward calculations. Here we have set

$$
\boldsymbol{R}''_p = \frac{e}{m\Omega'}\int_{t'}^{t''} \sin\Omega'(t''-t)\,\mathrm{e}^{-J\omega(t''-t)/2}\boldsymbol{E}_\perp(t)\,\mathrm{d}t.
$$

From the Lagrangian (3), the equation of motion is of the form

$$
\ddot{\boldsymbol{r}}_\perp + \omega J\dot{\boldsymbol{r}}_\perp + \Omega^2\boldsymbol{r}_\perp = e\boldsymbol{E}_\perp(t)/m. \tag{14}
$$

The classical path of (14) can be written as

$$
\begin{aligned}
\boldsymbol{r}^{\mathrm{cl}}_\perp(t) &= \boldsymbol{R}_p(t) + [\sin\Omega'(t''-t)\,\mathrm{e}^{-J\omega(t-t')/2}\boldsymbol{r}'_\perp + \sin\Omega'(t-t') \\
&\qquad \times \mathrm{e}^{J\omega(t''-t)/2}(\boldsymbol{r}''_\perp - \boldsymbol{R}''_p)]/\sin\Omega' T
\end{aligned}
\tag{15}
$$

where

$$
\boldsymbol{R}_p(t) = \frac{e}{m\Omega'}\int_{t'}^{t} \sin\Omega'(t-\tau)\,\mathrm{e}^{-J\omega(t-\tau)/2}\boldsymbol{E}_\perp(\tau)\,\mathrm{d}\tau.
$$

At caustics $\Omega' T = j\pi$, (15) is invalid unless the following condition is satisfied:

$$
\sin\Omega'(t''-t)\,\mathrm{e}^{-J\omega(t-t')/2}\boldsymbol{r}'_\perp + \sin\Omega'(t-t')\,\mathrm{e}^{J\omega(t''-t)/2}(\boldsymbol{r}''_\perp - \boldsymbol{R}''_p) = 0. \tag{16}
$$

For our choice of $t$ in integrating (12), (16) reduces to the conditions of the arguments of two-dimensional Dirac $\delta^{(2)}$ appearing in (15) to vanish. In other words, there exists an infinite number of classical paths between the initial position $(\boldsymbol{r}'_\perp, t')$ and the final position $(\boldsymbol{r}''_\perp, t'')$. For the case of a constant magnetic field only, (13) reduces to

$$
K_\perp(\boldsymbol{r}''_\perp, \boldsymbol{r}'_\perp; T=2j\pi/\omega) = \exp(-\mathrm{i}j\pi)\delta^{(2)}(\boldsymbol{r}'_\perp - \boldsymbol{r}''_\perp) \tag{17}
$$

(the factor 2 in (24) of Cheng (1984b) should be removed) as we expect. Finally, we are able to obtain the propagator of our dynamical system by combining only (5) or (7) or (8) with (6) or (11) or (13) for different cases.

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